1 B totally bounded subset of metric space M

$\Rightarrow$

there exists a finite ε-net $S \subset M$ for B,

so for $x, y \in B$ we have ~~$|x-y| \leq$~~

$$d(x,y) \leq K \cdot 2\varepsilon,$$

with K the number of points in S,

so B is bounded ✓

bounded but not totally bounded:

$$\left\{ f_n : [-1,1] \to \mathbb{R} : f_n(x) = \begin{cases} 1 & x = \frac{1}{n} \\ 0 & \text{else} \end{cases}, \ n \in \mathbb{Z} \setminus \{0\} \right\}$$

✓

$\frac{8.5}{10}$ P.

2 a $\quad \mathcal{F}$ equicontinuous $\iff$ $\mathcal{F}$ compact

+ bdd + closed

$\frac{1}{2}$ b $\quad cl(F) = F \cup f \quad , \quad f(x) = \begin{cases} \delta(0) & x = 1 \\ 0 & x \in [0,1) \end{cases}$

$f(x) = 0,\ x \in [0,1]$

$f_n(x) \leq$ [illegible] $\frac{1}{n}$

c $\quad cl(F)$ compact in $C[0,1]$ if

$Cl(F)$ equicontinuous in $C[0,1]$:

$\forall \varepsilon > 0 \ \exists \delta$ such that $\forall x, y \in [0,1]$

$d(x,y) < \delta \Rightarrow d(f(x), f(y)) < \varepsilon \qquad \forall f \in Cl(F)$

$d(\delta(0), 0)$ is never smaller than any $\varepsilon > 0$,
so $Cl(F)$ is not equicontinuous in $C[0,1]$
and therefore not compact in $C[0,1]$

but wrong.

3 a $\quad \emptyset \in \mathcal{J}$ and $\mathbb{R} \setminus \mathbb{R}$ finite $\Rightarrow \mathbb{R} \in \mathcal{J}$ $\quad$ (T1)

$\frac{2}{2,5}$

$U_1 \in \mathcal{J}$ and $U_2 \in \mathcal{J}$

$\Rightarrow$

$\mathbb{R} \setminus U_1$ finite and $\mathbb{R} \setminus U_2$ finite

$\Rightarrow$

$(\mathbb{R} \setminus U_1) \cup (\mathbb{R} \setminus U_2) = \mathbb{R} \setminus (U_1 \cap U_2)$ finite

$\Rightarrow$

$(U_1 \cap U_2) \in \mathcal{J}$ $\quad$ (T2)

$U_i \in \mathcal{J} \quad \forall i \in I$

$\Rightarrow$

$\mathbb{R} \setminus U_i$ finite $\quad \forall i \in I$

$\Rightarrow$

$\bigcap_{i \in I} (\mathbb{R} \setminus U_i) = \mathbb{R} \setminus \left(\bigcup_{i \in I} U_i \right)$ finite

$\Rightarrow$

$\left(\bigcup_{i \in I} U_i \right) \in \mathcal{J}$ $\quad$ (T3)

b $\quad (\mathbb{R}, \mathcal{J})$ is compact if we can find a finite cover for it (using sets from $\mathcal{J}$)

sub

only cover !

lets take $U_1 \in \mathcal{J}$

if U_1 is in the cover, ~~we need~~ then $\mathbb{R} \setminus U_1$ is left to cover

and $\mathbb{R} \setminus U_1$ is finite, so we can cover it with finite sets from $\mathcal{J}$.

So we can cover $(\mathbb{R}, \mathcal{J})$ with U_1 unified with a finite amounts of sets from $\mathcal{J}$

$\Rightarrow$ there exists finite open subcover for $(\mathbb{R}, \mathcal{J})$

$\Rightarrow$ $(\mathbb{R}, \mathcal{J})$ compact

3 ~~a~~ Suppose $(\mathbb{R}, \mathcal{J})$ is not Hausdorff,

then $\forall x, y \in \mathbb{R}$ with $x \neq y$,

U and V open, $x \in U$, $y \in V \Rightarrow U \cap V \neq \emptyset$

so $x \in U \Rightarrow y \in U$

Take $U_i = \mathbb{R} \setminus \{i\}$ $i \in \mathbb{R}$

$(\mathbb{R} \setminus U_i = \{i\}$ finite $\Rightarrow U_i$ open$)$

We know that $x \in U_i$ $\forall i \neq x$,

then also $y \in U_i$ $\forall i \neq x$.

$$\Rightarrow y \in \bigcap_{i \in \mathbb{R}\setminus\{x\}} U_i = \bigcap_{i \in \mathbb{R}\setminus\{x\}} (\mathbb{R} \setminus \{i\}) = \mathbb{R} \setminus \bigcup_{i \in \mathbb{R}\setminus\{x\}} \{i\}$$

$$= \mathbb{R} \setminus (\mathbb{R} \setminus \{x\}) = \{x\} \Rightarrow x = y \quad ↯$$

So $(\mathbb{R}, \mathcal{J})$ is Hausdorff.

d Suppose $(\mathbb{R}, \mathcal{J})$ is not connected,

then $\exists f : \mathbb{R} \to \{0,1\}$ continuous

$\Rightarrow$ $f^{-1}(0)$ ~~open/is~~ $\in \mathcal{J}$ and $f^{-1}(1) \in \mathcal{J}$ ~~op~~

We also know that $f^{-1}(1) = \mathbb{R} \setminus f^{-1}(0)$

and this ~~is~~ is finite because $f^{-1}(0) \in \mathcal{J}$.

Which means $\mathbb{R} \setminus f^{-1}(1)$ ~~\$~~ can not be finite.

But from $f^{-1}(1) \in \mathcal{J}$ it follows that $\mathbb{R} \setminus f^{-1}(1)$ ~~¢~~ is finite.

So we have a contradiction here.

$\Rightarrow (\mathbb{R}, \mathcal{J})$ is connected.

5 c Suppose $(\mathbb{R}, \mathcal{J})$ is Hausdorff,

then for $x, y \in \mathbb{R}$, $x \neq y$, there exist U and V open with $x \in U$, $y \in V$, $U \cap V = \emptyset$.

We know that $\mathbb{R} \setminus U$ and $\mathbb{R} \setminus V$ are finite.

$$U \cap V = \emptyset \Rightarrow V \subset \mathbb{R} \setminus U$$

$$\Rightarrow V \text{ is finite}$$

$$\Rightarrow \mathbb{R} \setminus V \text{ is not finite} \quad ↯$$

$\Rightarrow (\mathbb{R}, \mathcal{J})$ is not Hausdorff

e $\mathbb{R} \setminus U$ is finite $\forall U \in \mathcal{J}$,
so each $U \in \mathcal{J}$ contains different holes,
therefore we can not compute distances between points in $\mathbb{R}$? a bit vague
$\Rightarrow$
$\mathcal{J}$ can not be generated by a (non-Euclidean) metric defined on $\mathbb{R}$

4 a $Cl(\mathbb{R}) = \mathbb{R}$, $Cl(\mathbb{R}\setminus\mathbb{R}) = \emptyset$

$\frac{2.5}{2.5}$

$b(\mathbb{R}) = \mathbb{R} \cap \emptyset = \emptyset$

$\mathbb{R}$ is dense in $\mathbb{R}$

b $Cl(\mathbb{Q} \cap [-1,2)) = [-1,2] \neq \mathbb{R}$

$\Rightarrow$ $\mathbb{Q} \cap [-1,2)$ not dense

$Cl(\mathbb{R}\setminus(\mathbb{Q} \cap [-1,2))) = \mathbb{R}$

$b(\mathbb{Q} \cap [-1,2)) = [-1,2] \cap \mathbb{R} = [-1,2]$

$B_1(0) \in [-1,2] \Rightarrow \mathbb{Q} \cap [-1,2)$ not nowhere dense

c $Cl(\{\frac{1}{n} : n \in \mathbb{Z}, n \neq 0\}) = \{0\}$ $\cup \{\frac{1}{n} : \dots\}$

$\Rightarrow$ not dense , $\text{Int}\{0\} = \emptyset \Rightarrow$ nowhere dense

$Cl(\mathbb{R}\setminus\{\frac{1}{n} : n \in \mathbb{Z}, n \neq 0\}) = \mathbb{R}$

$b(\{\frac{1}{n} : n \in \mathbb{Z}, n \neq 0\}) = \{0\} \cap \mathbb{R} = \{0\}$

d $Cl(\mathbb{R}\setminus\mathbb{Q}) = \mathbb{R} \Rightarrow \mathbb{R}\setminus\mathbb{Q}$ dense in $\mathbb{R}$

$Cl(\mathbb{R}\setminus(\mathbb{R}\setminus\mathbb{Q})) = \mathbb{R}$

$b(\mathbb{R}\setminus\mathbb{Q}) = \mathbb{R} \cap \mathbb{R} = \mathbb{R}$

e $Cl(\mathbb{R}\setminus\mathbb{Z}) = \mathbb{R}$, $\mathbb{R}\setminus\mathbb{Z}$ dense in $\mathbb{R}$

$Cl(\mathbb{R}\setminus(\mathbb{R}\setminus\mathbb{Z})) = \mathbb{Z}$

$b(\mathbb{R}\setminus\mathbb{Z}) = \mathbb{R} \cap \mathbb{Z} = \mathbb{Z}$

f $Cl(\mathbb{Z}) = \mathbb{Z} \Rightarrow \mathbb{Z}$ not dense in $\mathbb{R}$

$\text{Int}\,\mathbb{Z} = \emptyset \Rightarrow \mathbb{Z}$ nowhere dense

$Cl(\mathbb{R}\setminus\mathbb{Z}) = \mathbb{R}$

$b(\mathbb{Z}) = \mathbb{Z} \cap \mathbb{R} = \mathbb{Z}$